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STRUCTURAL OPTIMIZATION USING THE ENERGY METHOD WITH INTEGRAL MATERIAL BEHAVIOUR

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Abstract. *With the advances of the computer technology, structural optimization has become a prominent field in structural engineering. In this study an unconventional approach of structural optimization is presented which utilize the Energy method with Integral Material behaviour (EIM), based on the Lagrange's principle of minimum potential energy. The equilibrium condition with the EIM, as an alternative method for nonlinear analysis, is secured through minimization of the potential energy as an optimization problem. Imposing this problem as an additional constraint on a higher cost function of a structural property, a bilevel programming problem is formulated. The nested strategy of solution of the bilevel problem is used, treating the energy and the upper objective function as separate optimization problems. Utilizing the convexity of the potential energy, gradient based algorithms are employed for its minimization and the upper cost function is minimized using the gradient free algorithms, due to its unknown properties. Two practical examples are considered in order to prove the efficiency of the method. The first one presents a sizing problem of I steel section within encased composite cross section, utilizing the material nonlinearity. The second one is a discrete shape optimization of a steel truss bridge, which is compared to a previous study based on the Finite Element Method.*

1 INTRODUCTION

Structural optimization has gained considerable attention in the design of structural engineering structures, especially in the preliminary phase. The application of optimization in design would enable optimal and efficient shape of structures and to utilize properties which are not feasible with conventional design techniques. In structural optimization a cost function is minimized under design and equilibrium constrains. Mechanical systems are typically formulated by partial differential equations describing the equilibrium, compatibility and constitutive relations. Alternatively the state of equilibrium can be also formulated by a variational formulation, in which a variation of a given functional with respect to certain state variable is zero. Conventionally the equilibrium condition is secured by the Finite Element Method (FEM). In this work the Energy method with Integral Material behaviour (EIM) is employed, which ensures equilibrium through minimization of the potential energy. With an additional cost function, this constitutes a bilevel optimization problem. The outline of this paper is the following: initially a brief outline of the EIM is given, followed by formulation of the structural optimization and finally the method is applied on two practical examples.

2 STRUCTURAL OPTIMZIATION USING EIM

2.1 Formulation of the EIM

Lagrange's theorem of minimum of potential energy is a variational principle in which the sum of the internal, Π_i and external energy, Π_e is minimized with respect to a state variable, and it represents the fundamental principle of the EIM:

$$\Pi = \Pi_i + \Pi_e \rightarrow min. \tag{1}$$

In the latter formulation the equilibrium, compatibility and constitutive relations are incorporated; therefore they should be represented accordingly with respect to a certain state variable. In case of formulation on cross section level this is the deformation vector containing the strain ε_0 at the origin and the two curvatures and $\boldsymbol{\varepsilon} = [\varepsilon_0,\ \kappa_y, \kappa_z]^{\ T}$. The constitutive law is described using the integral description of the material, introduced by Raue in [1]. This is obtained by integration over the uniaxial stress-strain relationship resulting in the specific strain energy W, the F and Φ which describe the same behaviour of one specific material. The latter two functions are used within the strain integration over complex geometries, in order the internal potential energy to be obtained:

$$W = W(\varepsilon) = \int_0^{\varepsilon} \sigma(\varepsilon) d\varepsilon, \quad F = F(\varepsilon) = \int_0^{\varepsilon} W(\varepsilon) d\varepsilon, \quad \Phi = \Phi(\varepsilon) = \int_0^{\varepsilon} F(\varepsilon) d\varepsilon. \tag{2}$$

Taking into account Bernoulli's hypothesis, the strain at arbitrary point of a deformed cross section could be described by a linear function of ε_0, κ_y and κ_z with respect to y and z coordinates respectively as:

$$\varepsilon_x(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z. \tag{3}$$

In case of biaxial bending, there is a second system of Cartesian coordinates η and ζ, at which along the η axis, the strain is constant as displayed on Figure 1. Here, standard relations are employed for the transformation between coordinate systems. The strain energy Π_i^C of a cross section with area A can be obtained by integrating the specific strain energy over the area $W(y,z)$:

$$\Pi_i^C = \iint_A W[\varepsilon(y,z)] dydz = \oint_L -\frac{\kappa_z}{\kappa^2} F dy + \frac{\kappa_y}{\kappa^2} F dz = -\frac{1}{\kappa} \oint_L F d\eta. \tag{4}$$

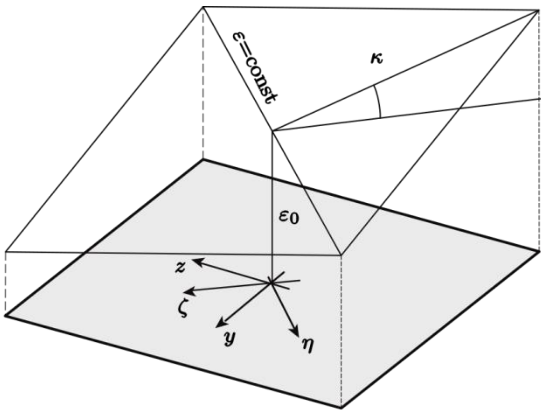

Figure 1. Coordinate transformation.

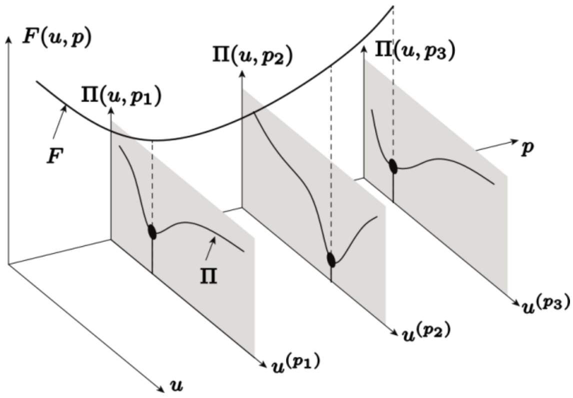

Figure 2. Bilevel optimization.

The proof of equation (4) is obtained by taking the divergence of the gradient of the $\boldsymbol{\Phi}(\boldsymbol{y},\boldsymbol{z})$ function, which is a 2-dimensional vector field, $\nabla \cdot \nabla\boldsymbol{\Phi}$ and using Green's divergence theorem to transfer the integral over the area into a line integral over the contour L. Within this study, the proof will not be presented; however it is discussed in detail in [1] along with the proof of the curl of the gradient $\nabla \times \nabla\boldsymbol{\Phi}$=0, which ensures the conservation of the energy. The external energy as a result of external N, M_y, M_z forces acting on a cross section is obtained as:

$$\Pi_e^C = -\left(N\varepsilon_0 + M_y\kappa_z + M_z\kappa_y\right). \quad (5)$$

With the last relation, equation (1) is completed and with unconstrained optimization algorithm the energy is minimized with the deformation vector $\boldsymbol{\varepsilon}$ as unknown variable. In case of element beam formulation, the internal energy has to be integrated over the length l:

$$\Pi_i^E = \int_0^l \Pi_i^C(x)dx. \quad (6)$$

The state variable in this case is the displacement vector $\boldsymbol{u}(x) = [u\ v\ w\ v'w'\ ]^T$, which is composed of the displacements u, v, w in x, y and z direction respectively and their first derivatives v' and w' with respect to x, representing the rotations. The relation between the deformation and displacement vector for geometrically linear Bernoulli beam is defined by the compatibility conditions:

$$\varepsilon_0 = u',\ \kappa_y = -v'',\ \kappa_z = -w''\ . \quad (7)$$

By integrating the product of the external force vector $\boldsymbol{p}(x) = [p_x\ p_y\ p_z\ m_y\ m_z\ ]^T$ and displacement vector $\boldsymbol{u}(x)$ over the length, the external energy yields into:

$$\Pi_e^E = \int_0^l \boldsymbol{p}^T(x)\boldsymbol{u}(x)\,dx. \quad (8)$$

The numerical implementation, discretization and suitable shape functions are discussed in the aforementioned literature.

2.1 Formulation of the structural optimization problem

A general structural optimization problem is formulated in a way to minimize an objective function which usually in mechanical problems represents the weight, the displacements, or the cost of production. Constraints imposed are the behavioural constraints with respect to the state variable (vector representing the response of a structure), design constraints on the design variable (vector or function describing geometry or material properties) and equilibrium constraints [3]. The behavioural and design constraints can relate to bounds, equality and inequality constraints with respect to mathematical optimization, while the equilibrium constraints are usually equality constraints. The EIM secures the equilibrium through optimization and therefore a Bilevel Optimization Problem (BOP) is formulated in which the minimization of the potential energy $\boldsymbol{\Pi}(\boldsymbol{u},\boldsymbol{p})$ represents the lower objective function and an additional cost function F$(\boldsymbol{u},\boldsymbol{p})$:

$$BOP: \begin{cases} \min\limits_{\boldsymbol{u},\boldsymbol{p}} F(\boldsymbol{u},\boldsymbol{p}) \\ s.t. \begin{cases} G(\boldsymbol{u},\boldsymbol{p}) \leq \mathbf{0}, \\ \min\limits_{\boldsymbol{u}} \Pi(\boldsymbol{u},\boldsymbol{p}), \end{cases} \end{cases} \tag{8}$$

where $\mathrm{F}, \Pi: R^m \times R^n \to R$; and $\mathrm{G}: R^m \times R^n \to R^l$. The design vector $\boldsymbol{p}$ contains the design variables in the upper optimization task, while the displacement vector $\boldsymbol{u}$ represents the state variables. The behavioural and design constraints are defined by a set of functions $G(\boldsymbol{u},\boldsymbol{p})$. Figure 2 depicts a simple nested bilevel optimization problem, with one state variable u and one design variable p, where the lower problem is only represented at sequences $p_{1,}$ $p_{2,}$ and p_3. In case of cross section optimization, the deformation vector would replace the displacement vector as a state variable. There is a vast application field of the bilevel programming problem; thus, the solution strategies depend on the properties of the lower and upper objective function. In this case the nested method is used, which deals with both optimization problems separately i.e. for each iteration of the upper objective, a separate optimization task is solved for the lower objective function. Assuming geometrically linear and unlimited deformation capacity, the potential energy might be considered as convex and smooth function; therefore, for its minimization the unconstrained gradient methods are very effective. In this case the Broyden-Fletcher-Goldfarb-Schano (BFGS) method is used from the Quasi-Newton algorithms with a line search for step size control from the Matlab Optimization toolbox. The properties of the upper objective are usually unknown, therefore gradient free, deterministic and stochastic represent a good choice. Here, the deterministic Nelder-Mead downhill simplex algorithm and the stochastic genetic algorithm from the evolutionary computing field are utilized. It should be noted that there exists a special class of problems which include variational inequality, i.e. the Mathematical Programs with Equilibrium Constraints (MPEC). However, these were not used in this case and the formulation using MPEC with EIM is a further research topic.

3 APPLICATION

3.1 Composite column

Composite cross sections using steel profile are commonly used in practice due to their efficiency to withstand high loads with relatively small area. This example presents optimization of completely encased steel I profile cross section by concrete with circular form typical for columns. The problem is formulated to compute the rotated shape of the I section with respect to the bending moment axes under biaxial bending and axial force, for minimum thickness of the flange t_w of the steel profile. The section geometry with the external forces is depicted on Figure 3 (left). The materials used in this case were concrete C30/37 with parabolic rectangular material law according to Eurocode 2 and steel S235 with bilinear constitutive law. The objective function was to minimize t_w, by changing the design variables t_w and α with constraints imposed on the strain in compression in concrete ε_c≥-3.5‰ and on the strain of the steel $|\varepsilon_c|$≤25‰. The contribution of the concrete in tension was neglected.

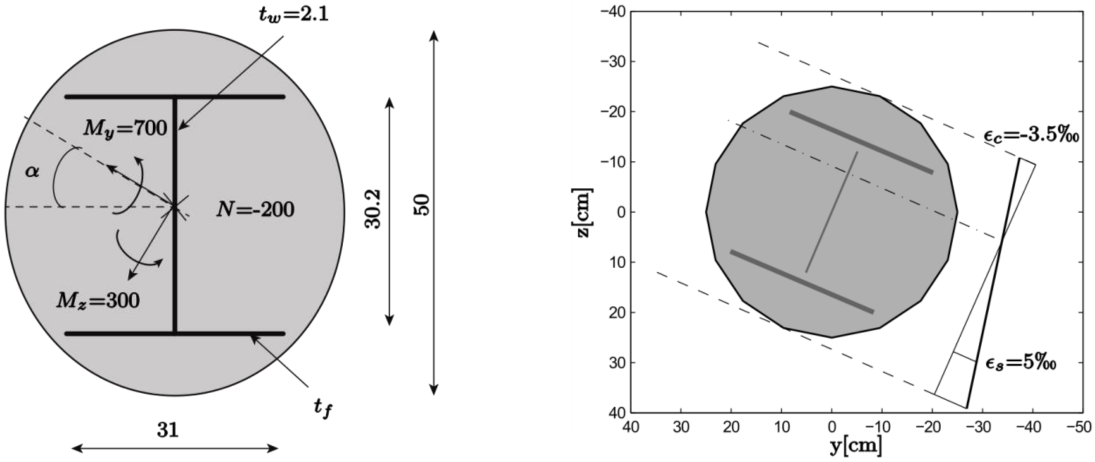

Figure 3. Circular section (left; units: [cm]; [kN]; kNm]). Computed curvature and optimized position (right).

The simplex algorithm was utilized and the constraints were imposed as penalty functions. The optimized cross section is depicted on Figure 3 (right), for which t_w=2.61cm and α=23.13°. As expected, the I profile rotated so it its web is perpendicular to the neutral axis with failure in compression.

3.2 Truss bridge

The second example presents discrete shape optimization and a sizing problem. It is based on a previous study by Soh & Yang in [2] who compared a solution of a truss bridge with previous work. Shape optimization problem has been identified as more difficult but more important task than mere sizing problems, since the potential savings in material can be far better improved by the latter [2]. The structure is a 24m spanned truss bridge depicted in Figure 4, for which the total weight should be minimized. Design variables are the area of the bars A_i (i =1, 2, …, 5), horizontal coordinates x_2, x_3, x_6, x_7 and vertical coordinates z_7, z_8. Constraints are imposed on the vertical and horizontal displacements (u <1cm, w <5cm), on the axial stresses (σ <14 kN/cm^2) and on the area of the bars (A_i >0,5 cm^2). The bridge is modelled with truss elements, using the symmetry. The material is considered as linear elastic with Young's modulus of E =2.1E6 kN/cm^2 and density ρ=7850 kg/m^3. Soh & Yang also include x_2 and x_3 as design variables without any notice of constraints, which resulted in this work with meaningless results as the node is moving to the support, thus the force has no influence. In order to compare the results, these variables were taken from the cited authors solution as fixed. For the optimization of the outer objective function, initially the GA was implemented, and after 200 generations with a population size of 20 individuals, the simplex was used to refine the results. Favorable results were obtained which resulted in weight reduction of 2.35%. The values of the design parameters are depicted in Table 1 and comparison of the shapes is displayed in Figure 5. The layout conform an arch which is close to the theoretical shape.

Study	Area [mm^2]					Coordinates [cm]						
	A_1	A_2	A_3	A_4	A_5	x_2	x_3	x_6	x_7	z_6	z_7	z_8
[3]	27.2	5136.5	106.7	1433.2	1420.8	162.2	579.3	167.3	435.0	581.2	184.1	61.4
Current	9.30	4676.5	389.8	1460.8	1475.9	fixed	fixed	176.5	433.4	604.9	158.6	38.4

Table 1. Truss bridge optimization results. Total weight: This work – 1235.6kg; Soh&Yang – 1265.32kg.

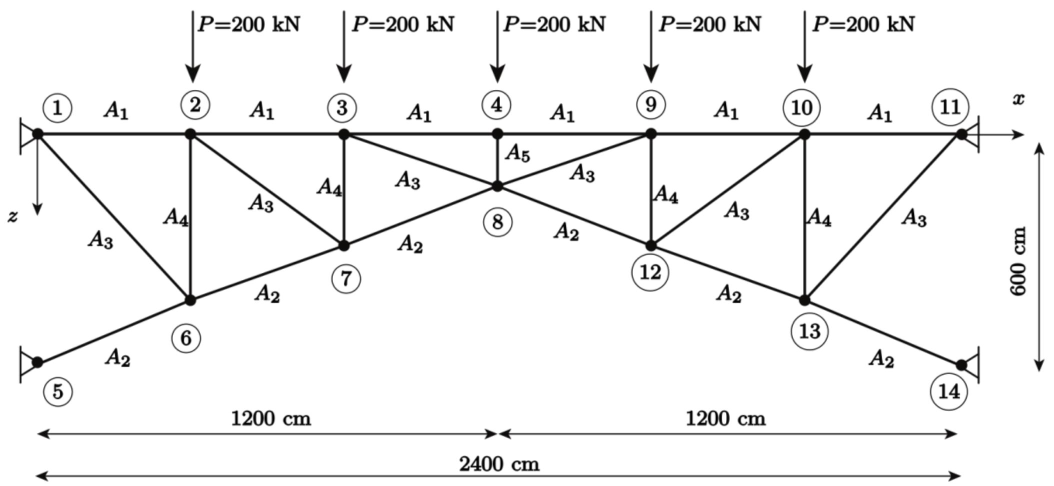

Figure 4. Truss bridge. Units: [cm]; [kN].

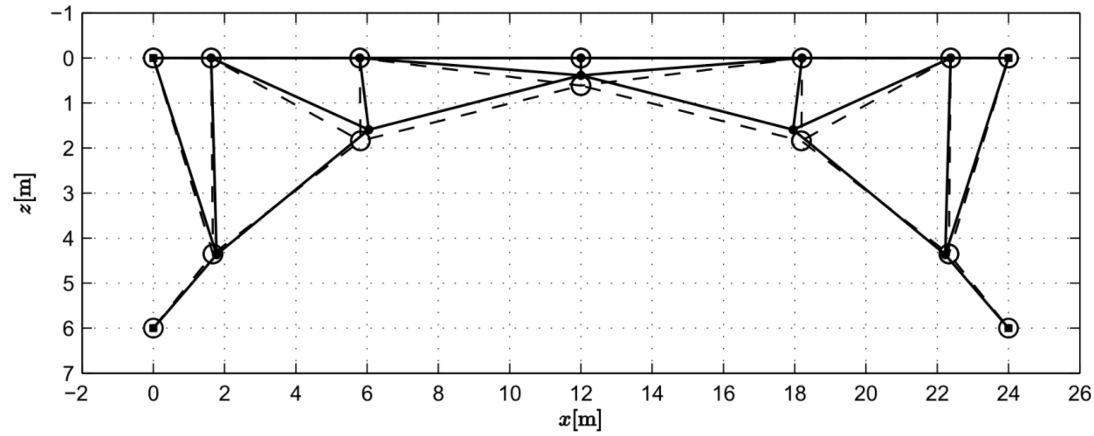

Figure 5. Comparison of truss bridge optimal shape: this work (-●-), Soh&Yang (- Θ -).

4 CONCLUSION

The future of design of attractive and efficient structures may be very closely related to structural optimization. In this paper, a structural optimization problem using EIM was formulated using bilevel optimization. Although computationally expensive, the EIM could be used as an alternative to the standard FEM for structural optimization, especially in case of physical nonlinearity. However, this limitation could be approached by formulating a MPEC.

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