Structural Optimization of Composite Structures using an Energy Method

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Abstract

The potential of structural optimization in the design of structures is growing rapidly with the advancement of the computer technology. In this paper, a formulation of structural optimization problem using an energy method is presented, for which solution a nested bilevel optimization is utilized. The method is applied on two practical examples, namely a nonlinear sizing problem for a two span girder and a discrete shape optimization of a truss bridge. These demonstrate the applicability of the method.

Keywords: Structural optimization; energy method; composite structures; truss optimization.

1 Introduction

Structural optimization has been widely applied in the industrial branches of engineering such as mechanical and aerospace engineering, due to importance of optimizing mass production items. The applications in the practical design of civil structures will likely also increase due to improvements in technology and methods. This would enable optimal and efficient shape of structures and to employ properties which are not feasible with conventional design techniques such as the redistribution of internal moments.

In structural optimization, a cost function is minimized under design and equilibrium constrains. Conventionally the equilibrium condition is secured through the application of the finite element method. However, in this work the *Energy method with Integral Material behaviour* (EIM) is used. The EIM ensures the equilibrium condition through minimization of the potential energy by a gradient based optimization procedure. The procedure is same for linear and nonlinear problem which is the asset of this method. Introducing an additional cost function a bilevel optimization problem is formulated. The nested optimization strategy will be employed to solve this problem which solves the two separate optimization tasks separately. Initially a brief introduction in the EIM will be given, followed by brief overview of the used optimization algorithms. Finally, the numerical implementation is discussed on two practical examples in order to discuss the applicability.

2 Formulation of the EIM

The energy method developed by Raue in [1] and [2] is based on two main principles: Lagrange's principle of minimum potential energy and the integral description of the material. The minimum of the potential energy Π, expressed by a sum of the internal Π_i and external Π_e, is secured with mathematical optimization and ensures equilibrium for linear or nonlinear structural problems:

$$\Pi = \Pi_i + \Pi_e \rightarrow min. \qquad (1)$$

The integral description of the material is derived only for 1D stress strain material law and therefore it finds its application for beam

elements. The assumptions which hold in this method are the following:

- Bernoulli's hypothesis applies.
- Shear and torsion are neglected.
- Loads are acting in the shear centre.
- A perfect bond is assumed between different materials in a cross section.

2.1 Integral description of material

The integral description is obtained by integrating the stress-strain law over the strains resulting in the specific strain energy W, the F and the Φ functions, which describe the same behaviour of one specific material. The latter two functions are used within the strain integration over complex geometries in order to obtain the internal potential energy:

$$W = W(\varepsilon) = \int_0^{\varepsilon} \sigma(\varepsilon) d\varepsilon, \tag{2}$$

$$F = F(\varepsilon) = \int_0^{\varepsilon} W(\varepsilon) d\varepsilon, \tag{3}$$

$$\Phi = \Phi(\varepsilon) = \int_0^{\varepsilon} F(\varepsilon) d\varepsilon. \tag{4}$$

2.2 Cross section formulation

With Bernoulli's hypothesis, the strain at an arbitrary point of a deformed cross section could be described by a linear function of the strain at the origin and the curvatures with respect to y and z coordinates as:

$$\varepsilon_x(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z. \tag{5}$$

In case of biaxial bending, a new coordinate system exists with coordinates η and ζ, at which along the η axis the strain is constant as shown in Figure 1. The relation between the two coordinate systems is described using the transformation

$$\eta = y\cos\varphi + z\sin\varphi = \frac{\kappa_z}{\kappa} y - \frac{\kappa_y}{\kappa} z, \tag{6}$$

$$\zeta = -y\sin\varphi + z\cos\varphi = \frac{\kappa_y}{\kappa} y + \frac{\kappa_z}{\kappa} z, \tag{7}$$

where φ represents the angle between the y and η axes, $\kappa^2 = \kappa_y^2 + \kappa_z^2$ represents the magnitude of the gradient in the transformed $\eta - \zeta$ coordinate system. Therewith, the strain can be represented only with respect to the ζ coordinate.

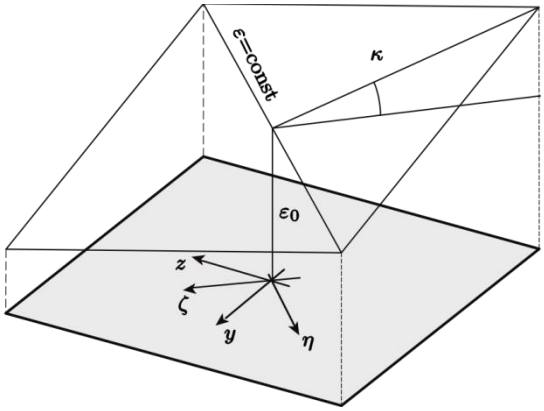

Figure 1. Coordinate transformation.

The strain energy Π_i^C of a cross section with area A can be obtained by integrating the specific strain energy over the area (y,z) :

$$\Pi_i^C = \iint_A W dA = \iint_A W[\varepsilon(y,z)] dy dz. \tag{8}$$

Raue in [1] used the Green's divergence theorem to transform this to a line integral over the contour L. First, with the gradient of the $\Phi(y,z)$, a 2-dimensional vector field $\mathbf{V}$ is formulated:

$$\mathbf{V} = \nabla\Phi = \begin{bmatrix} \frac{\partial\Phi}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial y} \\ \frac{\partial\Phi}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial z} \end{bmatrix} = \begin{bmatrix} \kappa_y F(y,z) \\ \kappa_z F(y,z) \end{bmatrix}. \tag{9}$$

The divergence of $\mathbf{V}(y,z)$ gives the relation between the $F(y,z)$ and the $W(y,z)$:

$$\nabla\cdot\mathbf{V} = \kappa_y \frac{\partial F}{\partial y} + \kappa_z \frac{\partial F}{\partial z} = \kappa^2 W(y,z). \tag{10}$$

In [2] it is proven that the curl is $\nabla\times\mathbf{V} = 0$, which ensures $\mathbf{V}$ to be a conservative field, representing conservation of the strain energy.

Substituting for the specific strain energy from equation (10) into (8) and using Green's theorem, the total strain energy is calculated as:

$$\Pi_i^C = \frac{1}{\kappa^2}\iint_A \left[\kappa_y \frac{\partial F}{\partial y} + \kappa_z \frac{\partial F}{\partial z}\right] dydz \\ = \oint_S -\frac{\kappa_z}{\kappa^2} F dy + \frac{\kappa_y}{\kappa^2} F dz = -\frac{1}{\kappa}\oint_L F d\eta. \tag{11}$$

The external energy is obtained from the forces N, M_y, M_z acting on the cross section:

$$\Pi_e^C = -\left(N\varepsilon_0 + M_y\kappa_z + M_z\kappa_y\right). \tag{12}$$

Having the external and internal energy, an unconstrained optimization problem is formulated with the total potential energy as an objective function, with the deformation vector $\boldsymbol{\varepsilon} = [\varepsilon_0,\ \kappa_y, \kappa_z]^T$ as unknown variable. The chosen algorithm for the optimization will be discussed in the following section. The solution at $\boldsymbol{\varepsilon}^* = [\varepsilon_0^*,\ \kappa_y^*,\ \kappa_z^*]^T$ presents the minimum potential energy i.e. the equilibrium of the system. Internal forces are calculated by taking the partial derivatives of the potential with respect to $\boldsymbol{\varepsilon}$.

2.3 Element formulation

The internal energy on element level is obtained by integration of the potential energy of a cross section over the length:

$$\Pi_i^E = \int_0^l \Pi_i^C(x) dx. \tag{13}$$

In this case the state variable (also the unknown parameter in the optimization) is displacements vector $\boldsymbol{u}(x) = [u\ v\ w\ v'\ w'\ ]^T$, where $u, v,$ and w are the displacements in $x, y,$ and z direction respectively and v', w' are the first derivatives with respect to x. The linear relationship between the displacements and deformation for a beam are given by the compatibility conditions:

$$\varepsilon_0 = u',\ \kappa_y = -v'',\ \kappa_z = -w''. \tag{14}$$

The external energy is obtained by integrating the product of displacements and the external forces $\boldsymbol{p}(x) = [p_x\ p_y\ p_z\ m_y\ m_z\ ]^T$ over the length:

$$\Pi_e^E = \int_0^l \boldsymbol{p}^T(x)\boldsymbol{u}(x)\, dx. \tag{15}$$

The total potential energy on element level Π^E is obtained by simple superposition of the internal and external energy on element level and with its minimization equilibrium is reached. If a system of elements is calculated, the potential energy is the sum of the energy from individual elements. The numerical implementation, as well as the discretization and the suitable shape functions reviewed in the aforementioned literature.

3 Structural optimization

A general structural optimization problem is formulated in a way to minimize an objective function which usually in mechanical problems represents the weight, the displacements, or the cost of production. Constraints imposed are the behavioural constraints with respect to the state variable (vector representing the response of a structure), constraints on the design variable (vector or function describing geometry or material properties) and equilibrium constraints [3]. As pointed out in the previous section, the EIM method secures the equilibrium through minimization of the potential energy. This constitutes a bilevel programming problem (BOP) where the minimization of the potential energy $\Pi(\boldsymbol{u}, \boldsymbol{p})$ represents the lower optimization task with respect to the displacement vector $\boldsymbol{u}$ as the state variable, and an additional objective function $F(\boldsymbol{u}, \boldsymbol{p})$ for optimizing the design variable vector $\boldsymbol{p}$ as the upper optimization task:

$$BOP: \begin{cases} \min\limits_{\boldsymbol{u},\boldsymbol{p}} F(\boldsymbol{u}, \boldsymbol{p}) \\ s.t. \begin{cases} G(\boldsymbol{u}, \boldsymbol{p}) \le \boldsymbol{0}, \\ \min\limits_{\boldsymbol{u}} \Pi(\boldsymbol{u}, \boldsymbol{p}), \end{cases} \end{cases} \tag{16}$$

where $F, \Pi: R^m \times R^n \to R$; and $G: R^m \times R^n \to R^l$. Here $G(\boldsymbol{u}, \boldsymbol{p})$ is a set of functions defining the behavioural and design constraints. The bilevel programming problem has a vast application field;

therefore, the solution strategies depend on the properties of the lower and upper objective function as well as the constraints. Colson in [4] gives a very good overview of the bilevel optimization problems. These involve branch-and-bound techniques in case of convex and upper and lower objective functions, penalty based approaches, trust-region, descent methods, nested optimization and evolutionary algorithms.

A special class of problems which includes variational inequality in the constraints are the *Mathematical Programs with Equilibrium Constraints* (MPEC), which formulates the upper objective function as a Lagrangian function by imposing the Karush-Kuhn-Tucker (KKT) conditions of the lower one, in case it is convex and differentiable. This is true for the energy in geometrically linear structures and in physically nonlinear problems with unlimited plastic deformation.

The methods solving a MPEC usually require also the upper function to be continuous and differentiable which for practical application is not always the case; therefore, in this case the nested method is used. This method solve the two optimization problems separately i.e. for each step of the upper optimization, the lower optimization is solved separately. Figure 2 depicts a simple nested bilevel optimization problem, with one state variable u and one design variable p, where the lower problem is only represented at sequences p_1, p_2, and p_3.

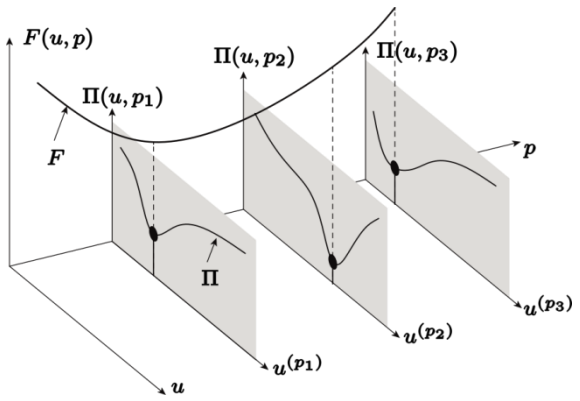

Figure 2. Bilevel optimization.

Utilizing the convexity of the energy, the Quasi-Newton methods present a good choice for the lower optimization problem. These methods tend to converge very fast for smooth convex functions as the search direction is determined by computing the gradient. In this case the Broyden-Fletcher-Goldfarb-Schano (BFGS) method was chosen for the approximation of the Hessian matrix, and the line search method for the step size control which guaranties a certain decrement of the objective function. Kelly in [5] gives a good overview of convex optimization.

As previously mentioned, often the properties of the upper objective function are unknown; therefore, the Evolutionary Algorithms (EA) present a good choice, especially in case of absence of an initial solution. Based on the natural selection from the evolution theory, the EA are stochastic, global and gradient free. In the initiation process, the function space is populated based on a probability distribution by initial population (parents) and the objective function (in this case the fitness function) is evaluated. Then the crossover and mutation processes take place among the parents by changing their binary form based on probabilistic information, and again the offspring are evaluated. Based on a certain survival rate, new population is formed from the fittest individuals. From the group of EA in this case, the Genetic Algorithm (GA) was used. Ebien gives an excellent introduction into the evolutionary computing in [6].

The computational cost of the evolutionary algorithm is usually very high since the fitness function needs to be evaluated each time at each individual and offspring, and the convergence is slow due to the probabilistic determination for next generation individuals. Therefore, in order to reduce computational time, a hybrid optimization technique was used to refine the search of the GA by using the Nelder-Mead downhill simplex algorithm. The simplex algorithm is a gradient free deterministic algorithm based on a simplex which is one dimension higher than the function space of the objective function. The properties of this algorithm are advantageous when not dealing with a smooth objective function. One disadvantage is the no-guarantee of convergence since it is not proven mathematically i.e. there could be stagnation at a non-optimal point; however, in practice their performance is

generally good [5] in case of a-priori knowledge of the optimization problem.

4 Numerical implementation and application

The previously described principles were implemented in a Matlab - based program using the optimization algorithms from the optimization toolbox. Figure 3 roughly outlines the optimization procedure, where the grey fields are the unknown parameters in the optimization process. The variable $\boldsymbol{p^o}$ is the parameter for each step of the upper optimization, for which the potential energy is minimized and $\boldsymbol{u^o}$ is obtained.

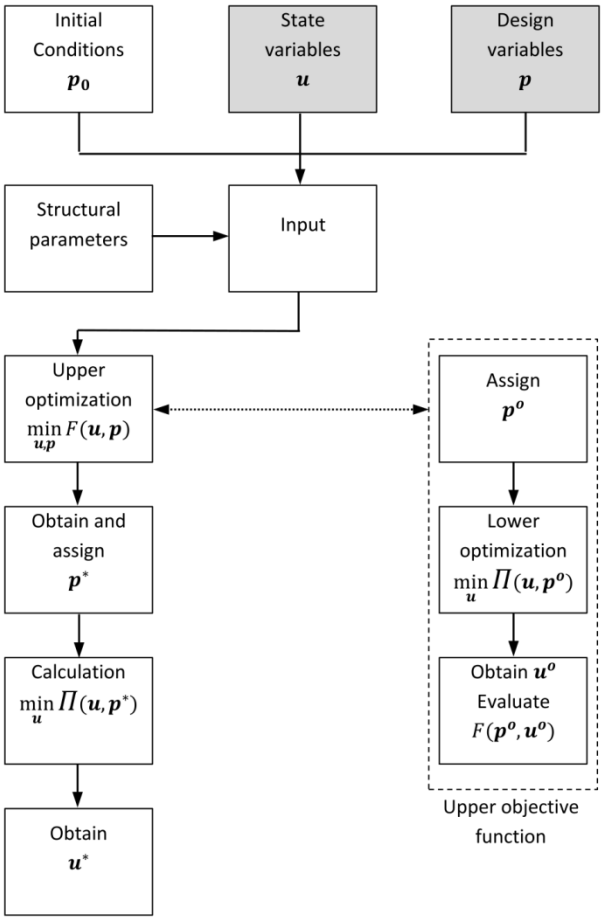

Figure 3. Nested optimization flowchart.

Structural optimization generally presents an iterative-intuitive procedure [3]; therefore, the initial conditions $\boldsymbol{p_0}$ have significant impact on the success of the optimization process. The calculation and optimization code was validated against benchmark examples; however, for the sake of brevity they will not be presented here.

4.1 Continuous girder

The first example is a sizing problem of a two span composite girder depicted in Figure 4, with composite cross section of I-steel profile and reinforcement slab on top subjected to dead and live load, for which the amount of reinforcement in the support has to be optimized. The idea is to keep a constant height of the cross section over the length and optimize the reinforcement at the support section to withstand the support moment, with respect to different limit states under the load combinations given in Table 1, taking into account the moment redistribution.

Table 1. Continuous girder load cases.

Load case	p_g [kN/m]	p_{q1} [kN/m]	p_{q2} [kN/m]
LC1	20	20	20
LC2	10	10	0
LC3	20	20	0

Concrete *C40/50* was used according to the Eurocode parabolic rectangular material law. The steel for the I-profile is S235, and B500 for the reinforcement using the bilinear material law, both including hardening after the yielding point. The girder is discretized into 8 beam elements of which the 4 near the support have the cross section with the reinforcement in the concrete slab. A total of 63 unknown parameters were used for the lower optimization. Initially the height I section was plastically designed according to the elastic span moment of LC3. Only the area of reinforcement A_s presents a design variable for the upper optimization; thus, the penalized simplex algorithm is a good choice. In Table 2 the optimization properties are listed.

Table 2. Continuous girder optimization properties.

Load case	LC1	LC2
Design variable	A_s	A_s
Objective ($\rightarrow min$)	A_s	A_s
Constraints	A_s >0 cm^2 $\lvert\varepsilon_s\rvert$ ≤25‰ ε_c ≥-3,5‰	A_s >0 cm^2 $\lvert\varepsilon_s\rvert$ ≤1‰ ε_c ≥-1‰ $u_{z,max}$ ≤10cm

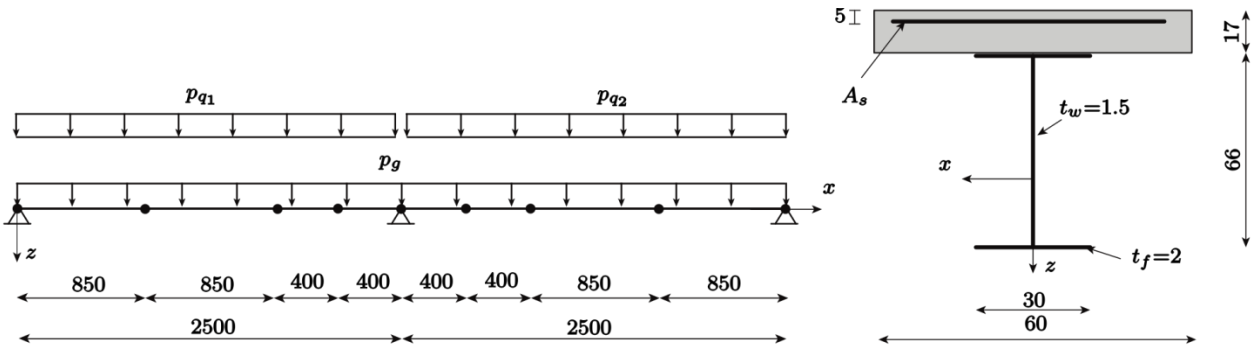

Figure 4. Continuous girder disposition (left) and composite cross section (right). Units: [cm].

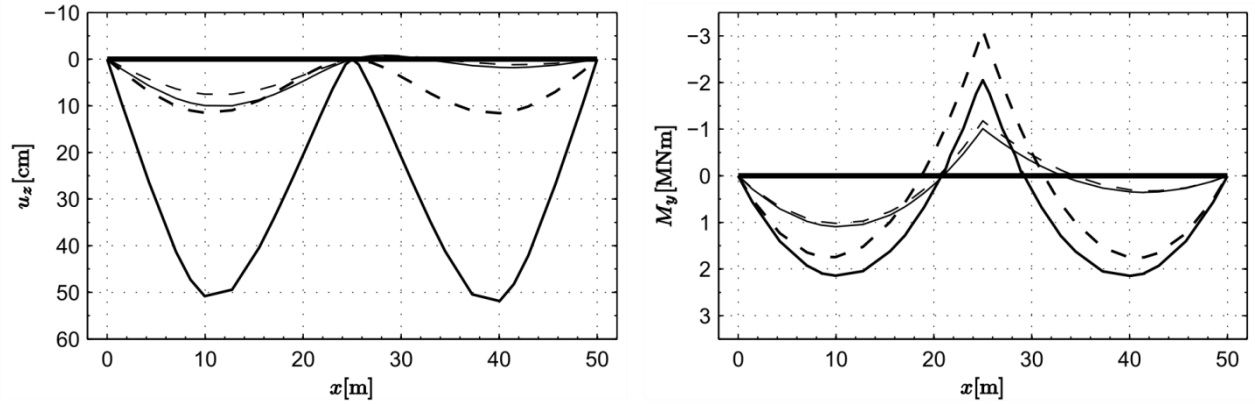

Figure 5. Vertical displacements (left) and bending moment (right): LC1 plastic (—), LC1 elastic (– –), LC2 plastic (—), and LC2 elastic (– –).

The constraints are taken as penalty functions. The optimization was done twice, once for LC1 and once for LC2. LC1 presents the ultimate limit state with higher load intensity, constrained to a local section failure, regardless of its position. LC2 has lower intensity and additional constraint.

Table 3. Continuous girder optimization results.

Variable	LC1	LC2
A_{pl} [cm]	13,26	16,18
$M_{y,min}^{el}$ [MNm]	-3,12	-1,18
$M_{y,max}^{el}$ [MNm]	1,75	1,03
$M_{y,min}^{pl}$ [MNm]	-2,05	-1,01
$M_{y,max}^{pl}$ [MNm]	2,15	1,09
u_z^{el} [cm]	11,58	7,53
u_z^{pl} [cm]	51,86	10,00

In Table 3 the results from the optimization are presented, where the required A_s for LC2 is 15% higher than LC1. The reduction of the support moment M_y due to moment redistribution is 34% and 14% for LC1 and LC2 and the span displacement is 77% and 24% higher, respectively. For comparison, the reinforcement area for elastic support moment for LC1 is 51,19 cm^2, which is 74% higher than the one optimized for the plastic. Figure 5 depicts the displacements and moment distribution and Figure 6 the convergence of LC2.

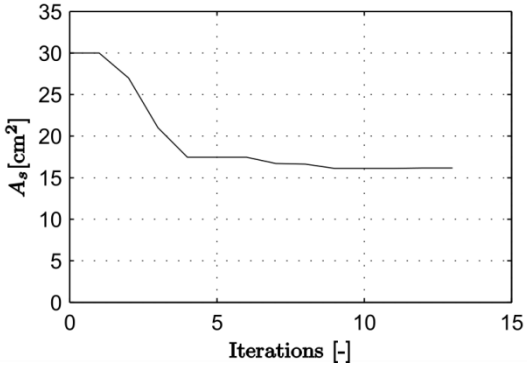

Figure 6. Convergence history LC2.

4.2 Truss bridge

The second example involves sizing and discrete shape optimization. It is based on previous work

by Soh & Yang in [7] who compared a solution of a truss bridge with previous work. A shape optimization problem has been identified as more difficult but more important task than mere sizing problems, since the potential savings in material can be significantly improved by the latter [7]. The structure presents a 24m spanned truss bridge as presented in Figure 7, for which the total weight should be minimized by sizing the area of the bars and choosing the optimal position of the nodal coordinates. The unknown parameters and constraints of the upper objective function are presented in Table 4.

Table 4. Truss bridge optimization properties.

Design variables	A_1, A_2, A_3, A_4, A_5, x_6, x_7, z_6, z_7, z_8
Objective ($\rightarrow min$)	$V = \sum_{i=1}^{n_{bars}} L_i A_i$
Constraints	A_i >0,5 cm^2 u <1 cm w <5 cm σ <14 kN/cm^2

For modelling of the bridge the symmetry is employed by constraining the x direction of nodes 4 and 8 and applying only half of the force in node 4. The material is linear elastic with Young's modulus of E =2.1E6 kN/cm^2 with ρ=7850 kg/m^3. The elements are considered as truss elements. Soh & Yang also include the x_2 and x_3 as design variables without any notice of constraints, which resulted in this work with meaningless results as the node is moving to the support, thus the force has no influence. In order to compare the results, these variables were taken from the cited authors solution as fixed. The hybrid method for the upper optimization function was used. After 200 generations with population size of 20 individuals of the genetic algorithm, the simplex algorithm was used to refine the results. Favourable results were obtained as the weight was reduced by 2,35% as shown in Table 5. A comparison of the shapes is displayed in Figure 8. The layout conforms to an arch close to the theoretical shape. The diagonal rods in this case are with higher slope, by which the horizontal component of the force is greater, thus their area is larger. However, the shorter length of the bottom arch reduces the weight significantly. The computational time required for the optimization was 12h.

Table 5. Truss bridge optimization results.

Variable	Soh&Yang	This work
A_1 [mm^2]	27,17	9,30
A_2 [mm^2]	5136,46	4676,49
A_3 [mm^2]	106,68	389,84
A_4 [mm^2]	1433,24	1460,75
A_5 [mm^2]	1420,78	1475,85
x_2 [cm]	162,20	fixed
x_3 [cm]	579,30	fixed
x_6 [cm]	167,30	176,51
x_7 [cm]	435,00	433,43
z_6 [cm]	581,20	604,93
z_7 [cm]	184,10	158,56
z_8 [cm]	61,40	38,42
Weight [kg]	1265,32	1235,62

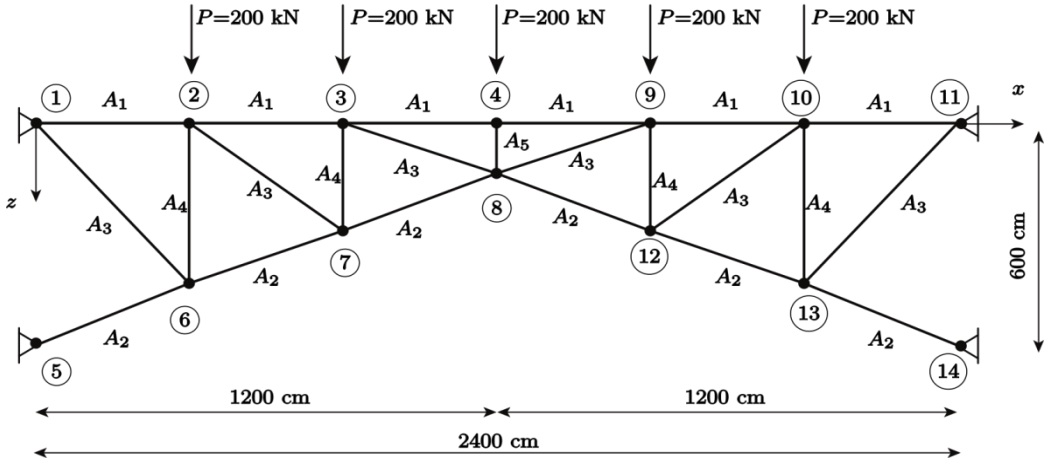

Figure 7. Truss bridge. Units: [cm]; [kN].

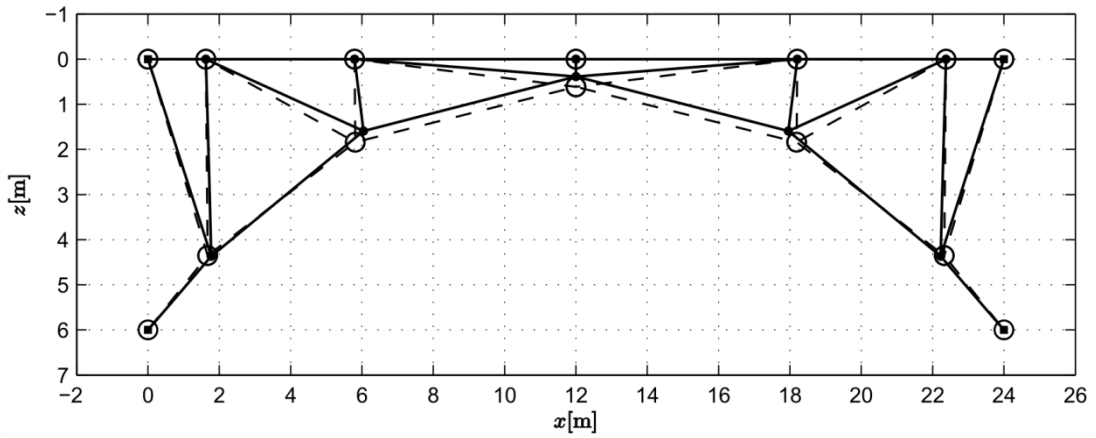

Figure 8. Comparison of truss bridge optimal shape: this work (-●-), Soh&Yang (- Θ -).

5 Conclusion

In this work a structural optimization problem was formulated as a bilevel optimization problem using an energy method. A nested method for the solution of the problem was implemented, constituted of two independent optimization algorithms, one for the energy as a lower objective function and the other for the upper cost function. The gradient based algorithms are a good choice for the potential energy, due to its continuity and convexity under the assumptions of geometrically linear and unlimited plastic deformations. For the upper objective function, gradient free algorithms were implemented i.e. the Nelder-Mead simplex and the genetic algorithm. The emphasis is put on the formulation of a structural optimization problem, rather than the choice of the algorithm. Using the EIM as equilibrium constraint is computationally more expensive compared to standard FEM optimization, especially for linear structures. This could be concluded from the computational time needed from the truss optimization problem, for which if a standard FEM equilibrium constrained optimization was used, the cost would be significantly lower for the same choice of algorithm. The reason is the lower nonlinear optimization that is performed each optimization step for obtaining the equilibrium in case of the EIM, which for the FEM would mean only an inversion of the stiffness matrix. The advantage using the proposed formulation would be in case of physically nonlinear problems, as the EIM would use the same procedure and the FEM needs additional iteration algorithm in the constrains for linearization of the energy such as the Newton-Raphson algorithm. Finally, as an outlook would be implementing a MPEC gradient based algorithm with which the upper cost function and the energy would be simultaneously minimized, increasing the computational efficiency.

6 References

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